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Statistical Entropy of Nonextremal Four-Dimensional Black Holes and U-Duality

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Abstract

We identify the states in string theory which are responsible for the entropy of near-extremal rotating four-dimensional black holes in $N = 8$ supergravity. For black holes far from extremality (with no rotation), the Bekenstein-Hawking entropy is exactly matched by a mysterious duality invariant extension of the formulas derived for near-extremal black holes states.

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Recent developments in string theory have led, for the first time, to an understanding of black hole entropy from a microscopic point of view. In [1] it was shown that the Bekenstein-Hawking entropy of an extremal, nonrotating, five-dimensional black hole precisely counts the number of BPS states in string theory with the given charges (in the limit of large charges). This agreement has since then been extended in a number of directions. Extreme 5D black holes with rotation [2], extreme 4D black holes [3], and slightly nonextreme 5D black holes [4,5,6] have all been shown to have a Bekenstein-Hawking entropy which agrees with the number of corresponding states in string theory. One goal of the present work is to show that this agreement continues to hold for slightly nonextremal 4D black holes.

The restriction to extreme or near-extreme black holes arises since we can only count states at weak coupling, while black holes only exist at strong coupling. For extremal configurations, one can argue that interactions are absent on the basis of supersymmetry and the extrapolation of the number of states from weak to strong coupling is justified. For near-extremal black holes there are situations in which the interactions are again suppressed. For black holes far from extremality, there appears to be no reason why a weakly coupled description is applicable.

Nevertheless, it was shown in [7] that there is a sense in which even black holes far from extremality can be thought of as composed of weakly interacting fundamental objects in string theory. The objects one needs are the same ones which yield the states of extremal black holes: extended solitons known as D-branes and fundamental strings. More precisely, ref. [7] considered a class of five-dimensional black holes labeled by the energy, three charges, and the asymptotic values of two scalars. The three charges are carried by onebranes, fivebranes and strings (or anti-branes, which are just branes with the opposite orientation and with the opposite sign of the charge). One can replace the original six parameters in the solution by the number of branes, anti-branes and strings ($N_1,\ N_{\bar{1}},\ N_5,\ N_{\bar{5}},\ n_R,\ n_L$) by matching the energy, three gauge charges and two scalar charges of these noninteracting objects with that of the black hole. In terms of these new variables the black hole entropy takes the suggestive form

$$S = 2\pi(\sqrt{N_1}+\sqrt{N_{\bar{1}}})(\sqrt{N_5}+\sqrt{N_{\bar{5}}})(\sqrt{n_L}+\sqrt{n_R}) \ . \tag{1}$$

This expression applies to all black holes, even those which are far from extremality. The symmetry of this expression is consistent with U-duality which permutes the three types of fundamental objects. It was argued [7] that (1) arises naturally from counting states in string theory, in the sense that it correctly reproduces the number of string states in three different weak-coupling limits, and is the simplest duality invariant expression with this property. However, no derivation of the general formula directly from counting states in string theory is currently available.

Since the significance of the above expression for the black hole entropy is not yet well understood, it is important to know whether it is special to five dimensions or if it applies more generally. In this paper we will show that the entropy of four-dimensional black holes can be expressed in a form directly analogous to (1). In [3] it was shown that the states of extremal four-dimensional black holes can be described in terms of D-twobranes, solitonic fivebranes, D-sixbranes, and open strings. We will consider the nonextremal version of

these solutions which is an eight parameter family of four-dimensional black holes. By comparing the mass, gauge charges, and scalar charges (which are pressures in the internal directions) of the black hole with those of a set of *noninteracting* branes and anti-branes, we will rewrite the Bekenstein-Hawking entropy in a form analogous to (1). We will show that in certain limits (corresponding to near-extremal black holes), the entropy formula we obtain indeed represents the number of states of this collection of branes at weak coupling.

The generalization of these black hole solutions to include rotation has recently been found [8]. We will show that in the limit of small rotation and near extremality the black hole entropy again agrees with the number of string states.

We will be considering Type II string theory compactified on $T^6 = T^4 \times S^1 \times \hat{S}^1$, which gives $N = 8$ supergravity in four dimensions. In ref. [9] general spherically symmetric black hole solutions of $N = 4$ supergravity in four dimensions were considered. Using these solutions it is straightforward to construct the general class of black holes in $N = 8$ supergravity. The starting point for this construction is a solution with four nonzero $U(1)$ gauge fields (carrying two electric and two magnetic charges) and three nontrivial scalars [9]. The Einstein metric is

$$\begin{aligned} ds^2 &= -f^{-1/2}(r)\left(1-\frac{r_0}{r}\right)dt^2 + f^{1/2}(r)\left[\left(1-\frac{r_0}{r}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2)\right] \ , \\ f(r) &= \left(1+\frac{r_0\sinh^2\alpha_2}{r}\right)\left(1+\frac{r_0\sinh^2\alpha_5}{r}\right)\left(1+\frac{r_0\sinh^2\alpha_6}{r}\right)\left(1+\frac{r_0\sinh^2\alpha_p}{r}\right) \ . \end{aligned} \tag{2}$$

This metric is parameterized by the five independent quantities α_2, α_5, α_6, α_p and r_0. The event horizon lies at $r = r_0$. The special case $\alpha_2 = \alpha_5 = \alpha_6 = \alpha_p$ corresponds to the Reissner-Nordström metric. The overall solution contains three additional parameters which are related to the asymptotic values of the three scalars. From the ten-dimensional viewpoint, these are the volume of the 4-torus $(2\pi)^4V$, and the radii of S^1 and $\hat{S}^1$, R_1 and R_2.

The physical charges are expressed in terms of these quantities as

$$\begin{aligned} Q_2 &= \frac{r_0 V}{g}\sinh 2\alpha_2 \ , \\ Q_5 &= r_0 R_2 \sinh 2\alpha_5 \ , \\ Q_6 &= \frac{r_0}{g}\sinh 2\alpha_6 \ , \\ n &= \frac{r_0 V R_1^2 R_2}{g^2}\sinh 2\alpha_p \ , \end{aligned} \tag{3}$$

where g is the ten-dimensional string coupling and we have chosen conventions such that $\alpha' = 1$ and the four-dimensional Newton constant is $G_4 = g^2/(8VR_1R_2)$. Note that in these conventions the string coupling is such that $g \to 1/g$ under S duality.

The ADM mass of the solution is

$$M = \frac{r_0 V R_1 R_2}{g^2}(\cosh 2\alpha_2 + \cosh 2\alpha_5 + \cosh 2\alpha_6 + \cosh 2\alpha_p) \tag{4}$$

and the Bekenstein-Hawking entropy is

$$S = \frac{A}{4G_4} = \frac{8\pi r_0^2 V R_1 R_2}{g^2} \cosh\alpha_2 \cosh\alpha_5 \cosh\alpha_6 \cosh\alpha_p \ . \tag{5}$$

There are three nontrivial scalar fields present in the solution and associated with these scalar fields are three pressures (scalar charges)

$$\begin{aligned} P_1 &= \frac{r_0 V R_1 R_2}{g^2}(\cosh 2\alpha_2 + \cosh 2\alpha_6 - \cosh 2\alpha_5 - \cosh 2\alpha_p) \ , \\ P_2 &= \frac{r_0 V R_1 R_2}{g^2}(\cosh 2\alpha_2 - \cosh 2\alpha_6) \ , \\ P_3 &= \frac{r_0 V R_1 R_2}{g^2}(\cosh 2\alpha_5 - \cosh 2\alpha_p) \ . \end{aligned} \tag{6}$$

In the ten-dimensional theory, the four charges (3) are carried by twobranes, fivebranes, sixbranes and strings. The D-sixbranes wrap around $T^4 \times S^1 \times \hat{S}^1$, the solitonic fivebranes wrap around $T^4 \times S^1$ and the D-twobranes wrap around $S^1 \times \hat{S}^1$. The strings carry momentum along the S^1 direction. In the spirit of [7] we calculate the values for the mass and scalar charges of each type of brane or string. This can be calculated from the solution we have presented by taking the four extremal limits: $r_0 \to 0$, $\alpha_i \to \pm\infty$ with Q_i and α_j $(j \neq i)$ fixed. We find that D-twobranes have mass and pressures

$$M = P_1 = P_2 = \frac{R_1 R_2}{g} \ , \qquad P_3 = 0 \ , \tag{7}$$

while for the sixbranes we have

$$M = P_1 = -P_2 = \frac{V R_1 R_2}{g} \ , \qquad P_3 = 0 \ . \tag{8}$$

For the solitonic fivebrane we have

$$M = -P_1 = P_3 = \frac{V R_1}{g^2} \ , \qquad P_2 = 0 \ , \tag{9}$$

and for the momentum we find

$$M = -P_1 = -P_3 = \frac{1}{R_1} \ , \qquad P_2 = 0 \ . \tag{10}$$

Using these relations plus the charges (3) we trade in the eight parameters of the solution for the eight quantities $(n_R, n_L, N_2, N_{\bar{2}}, N_5, N_{\bar{5}}, N_6, N_{\bar{6}})$ which are the numbers of right(left)-moving momentum modes, twobranes, anti-twobranes, fivebranes, antifivebranes, sixbranes and anti-sixbranes. We do this by matching the mass (4), pressures

(6), and gauge charges (3) with those of a collection of noninteracting branes. This leads to

$$\begin{aligned}
n_R &= \frac{r_0 V R_1^2 R_2}{2g^2} e^{2\alpha_p} \ , & n_L &= \frac{r_0 V R_1^2 R_2}{2g^2} e^{-2\alpha_p} \ , \\
N_2 &= \frac{r_0 V}{2g} e^{2\alpha_2} \ , & N_{\bar{2}} &= \frac{r_0 V}{2g} e^{-2\alpha_2} \ , \\
N_5 &= \frac{r_0 R_2}{2} e^{2\alpha_5} \ , & N_{\bar{5}} &= \frac{r_0 R_2}{2} e^{-2\alpha_5} \ , \\
N_6 &= \frac{r_0}{2g} e^{2\alpha_6} \ , & N_{\bar{6}} &= \frac{r_0}{2g} e^{-2\alpha_6} \ .
\end{aligned} \tag{11}$$

In terms of the brane numbers, the ADM mass is reexpressed as

$$M = \frac{1}{R_1}(n_R + n_L) + \frac{R_1 R_2}{g}(N_2 + N_{\bar{2}}) + \frac{V R_1}{g^2}(N_5 + N_{\bar{5}}) + \frac{V R_1 R_2}{g}(N_6 + N_{\bar{6}}) \ , \tag{12}$$

the gauge charges are simply differences of the brane numbers, and the other parameters are

$$V = \sqrt{\frac{N_2 N_{\bar{2}}}{N_6 N_{\bar{6}}}}, \qquad R_2 = \sqrt{\frac{N_5 N_{\bar{5}}}{g^2 N_6 N_{\bar{6}}}}, \qquad R_1^2 R_2 = \sqrt{\frac{g^2 n_R n_L}{N_2 N_{\bar{2}}}} \ . \tag{13}$$

The entropy (5) then takes the surprisingly simple form

$$S = 2\pi(\sqrt{n_R} + \sqrt{n_L})(\sqrt{N_2} + \sqrt{N_{\bar{2}}})(\sqrt{N_5} + \sqrt{N_{\bar{5}}})(\sqrt{N_6} + \sqrt{N_{\bar{6}}}) \ . \tag{14}$$

This is the analog of (1) for four-dimensional black holes. When one term in each factor vanishes, the black hole is extremal. In this case, (14) agrees with the number of bound states of these branes at weak coupling [3]. Although we cannot derive the general formula from counting string states, we can do so in certain limits corresponding to near-extremal black holes. Consider the case when $N_{\bar{2}} = N_{\bar{5}} = N_{\bar{6}} = 0$ and R_1 is large. We see from (12) that the lightest excitations will be the momentum modes. The extremal limit is obtained by also setting the number of left movers to zero $n_L = 0$. In that case the entropy can be calculated [3] as the entropy of a one-dimensional gas of $4N_2N_5N_6$ bosonic particles plus an equal number of fermionic particles with total energy $E = n_R/R_1$, which gives $S = 2\pi\sqrt{N_2N_5N_6n_R}$. In the near-extremal limit we also include left movers, which will be noninteracting if R_1 is large. Hence the entropy will be the sum

$$S = 2\pi\sqrt{N_2N_5N_6}(\sqrt{n_R} + \sqrt{n_L}) \tag{15}$$

which clearly agrees with (14) when $N_{\bar{2}} \sim N_{\bar{5}} \sim N_{\bar{6}} \sim 0$. Note that these antibrane excitations are very massive when R_1 is large, so one can see from (11), (12) that their number will be very small in the near-extremal limit and their contribution to the entropy will be negligible. We could do a similar calculation for the cases in which the lightest particles are the other branes. Since U-duality interchanges the different branes and strings, one expects a result similar to (15) with the indices permuted. Equation (14) is clearly the simplest duality invariant expression which agrees with these different nonextremal limits.

We have considered only four types of charges. Reducing Type II string theory to four dimensions on T^6 leads to a theory whose low energy limit is $N = 8$ supergravity. This contains 28 gauge fields and 70 scalars. The gauge fields can carry either electric or magnetic charges, so there are 56 possible charges. Each of these charges is carried by a different type of soliton in the ten-dimensional theory. From black hole uniqueness theorems [10] it is clear that the Bekenstein-Hawking entropy of the general solution depends on the energy and 56 "solution generating parameters" that add charge. However, these parameters are not the physically normalized charges, but also involve the asymptotic values of the scalars. From the special form of the coupling of scalars to gauge fields in $N = 8$ supergravity [11], one sees that a basis may be chosen for the scalars in which only 56 of them enter in the normalization of the gauge charges. One can view these parameters as 55 scalars and the total energy. The entropy can then be viewed as a function of 56+56 parameters which may be interpreted as the number of solitons and anti-solitons.

Since the full theory should be E_7 invariant we should be able to write the general entropy formula in an invariant way. If we denote by V_1^A the 56-dimensional vector giving the number of solitons and by V_2^A the number of anti-solitons, the formula for the entropy may take the form

$$S = 2\pi \sum_{i,j,k,l} \sqrt{T_{ABCD} V_i^A V_j^B V_k^C V_l^D} \ , \tag{16}$$

where T_{ABCD} is the quartic invariant considered in [12], where this formula was derived for the extremal case ($V_2^A = 0$).

We now consider adding rotation to the black holes discussed above. Since the rotation dependent terms in the solution fall off faster at infinity than the charges, the definition of the brane numbers (11) is unchanged. If we again take nearly extremal black holes with $N_{\bar{2}} \sim N_{\bar{5}} \sim N_{\bar{6}} \sim 0$, and R_1 large, the Bekenstein-Hawking entropy takes the form [8] [1]

$$S = 2\pi \left(\sqrt{n_R N_2 N_5 N_6} + \sqrt{n_L N_2 N_5 N_6 - J^2} \right) \ . \tag{17}$$

where J is the angular momentum of the black hole. This agrees precisely with the counting of string states as follows. With R_1 much larger than the other compact dimensions and with just twobranes and sixbranes present, the D-brane excitations of this system are described by a 1+1-dimensional field theory which turns out to be a $(4,4)$ superconformal sigma model [2]. The fivebrane breaks the right-moving supersymmetry [13], leaving us with $(0,4)$ superconformal symmetry. The $N = 4$ superconformal algebra gives rise to a left-moving $SU(2)$ symmetry. Since fermionic states in the sigma model become spinors in spacetime, the action of $O(3)$ spatial rotations has a natural action on this $SU(2)$. The charge F_L under one $U(1)$ subgroup of this $SU(2)$ will then be related to the four-dimensional angular momentum (along one of the three axes) carried by the left movers by $J = F_L/2$. Due to the presence of the fivebrane the right-moving $SU(2)$ symmetry of the original $(4,4)$ superconformal field theory is broken and the right movers cannot carry macroscopic angular momentum. The number of states with fixed n_L, n_R, $F_L \gg 1$ may

[1] There is a difference in the definition of J from [8], here we are measuring J in units of $\hbar$.

be computed as in [2,6] to yield the entropy

$$S = 2\pi\sqrt{\frac{c}{6}}(\sqrt{n_R} + \sqrt{\tilde{n}_L}) \ , \tag{18}$$

where $\tilde{n}_L = n_L - 6J^2/c$ is the effective number of left movers that one is free to change once one has demanded that we have a given macroscopic angular momentum. For our problem the central charge is $c = 6N_2N_5N_6$ [3], thus the entropy (17) agrees with the D-brane formula (18).

It is interesting to take the extremal limit of these rotating black holes, when the mass takes the minimum value consistent with given angular momentum and charges. This happens when $\tilde{n}_L = 0$, so the left movers are constrained to just carry the angular momentum and do not contribute to the entropy. When the angular momentum is nonzero, even the extremal black hole is not supersymmetric. Using (18) and writing the result in terms of the charge $n = n_R - n_L$ we find

$$S = 2\pi\sqrt{J^2 + nQ_2Q_5Q_6} \ , \tag{19}$$

which indeed agrees with the entropy of an extremal charged rotating black hole [8]. Notice the surprising fact that although we derived this formula in the large R_1 regime (and $J/M^2 \ll 1$), it continues to be valid for arbitrary values of the parameters. Since this is far from the BPS state, we had no reason to expect the weak-coupling counting to continue to agree with the black hole entropy.†

To summarize, we first considered four-dimensional nonrotating black holes. We argued that there is a sense in which one can view the general nonextremal black hole as composed of a collection of noninteracting branes and anti-branes. The number of branes of each type is determined by matching physical properties of the branes with those of the black hole. In terms of these numbers, the Bekenstein-Hawking entropy takes the simple form (14). We were able to show that in certain limits, this expression agrees with the number of states of this collection of branes and anti-branes at weak coupling. A complete derivation of these formula remains an outstanding challenge. We also showed that for nearly extremal *rotating* black holes, the entropy again agrees with the number of string states. Surprisingly, the extremal rotating black hole entropy was precisely matched by a D-brane counting argument, even far beyond the regime in which this counting was done.

There have been earlier indications [4] that the counting of string states at weak coupling agrees with the black hole entropy even in situations where one could not justify the extrapolation to strong coupling. We found another example of this in the case of extremal rotating black holes. The surprising success of these weak-coupling arguments indicates that understanding black hole entropy may be even simpler than it appears today.

† In five dimensions, taking the extremal limit of the results in [6] one obtains the microscopic entropy $S = 2\pi\sqrt{Q_1Q_5n + J_1J_2}$. This again agrees with the entropy of a black hole with two rotation parameters. This black hole is supersymmetric only when $J_1 = -J_2$ (and also for $J_1 = J_2$ with the opposite sign of one of the charges).

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